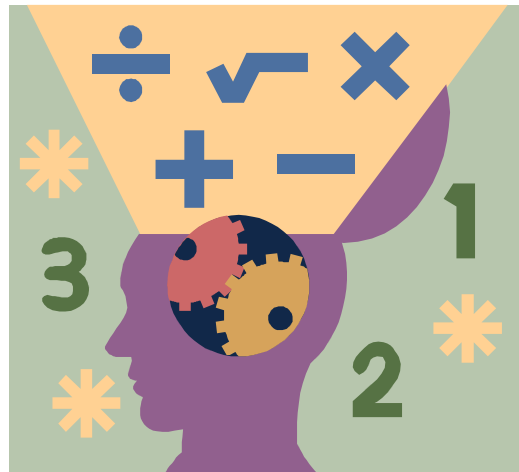


Clarifications of the Mathematics Standards



Have the Standards Changed?

The standards have been reorganized, but the language of the standards has remained the same. In 1995 when Delaware’s Mathematics Curriculum Frameworks was adopted, Delaware had ten standards. The first four focused on mathematical processes, and the next six focused on mathematical content. Delaware has now followed the lead of the National Council of Teachers of Mathematics (NCTM) and positioned the Content Standards before the Process Standards.

1995	Process Standards	2006
Standard 1	Problem Solving	Standard 5
Standard 3	Reasoning and Proof	Standard 6
Standard 2	Communications	Standard 7
Standard 4	Connections	Standard 8

In 1995, the writers of the Delaware standards felt that the distinction between computation and number sense and between algebra and patterns was sufficiently important to place them in separate standards. It is now time to bring those concepts back together. Computation has now been combined with number sense to form a new standard called Numeric Reasoning. Standards 7 and 10 have been combined to form a new standard called Algebraic Reasoning. Measurement has been pulled out of Standard 5 and moved to Geometric Reasoning. Statistics and Probability has been renamed Quantitative Reasoning.

1995	2005
Standard 5 Estimation, Measurement, and Computation	Standard 1 Numeric Reasoning
Standard 6 Number Sense	
Standard 7 Algebra	Standard 2 Algebraic Reasoning
Standard 10 Patterns, Relations, and Functions	
Standard 8 Spatial Sense and Geometry	Standard 3 Geometric Reasoning
Standard 9 Statistics and Probability	Standard 4 Quantitative Reasoning

What Does Research Say about Effective Instruction in Mathematics?

Students need to learn mathematics as a *living language* that is useful in describing and solving problems in the world. Language can only be learned well by using it in meaningful ways.

Student Learning

Research shows that learning mathematics deeply requires that students routinely engage in solving challenging problems. In order to maximize student opportunities to reason about important mathematical ideas, mathematics classrooms must provide plentiful opportunities for all students to talk, listen, and write about these ideas. “Students who...practice communication of mathematical concepts and procedures in concise, logical, and well-articulated ways—both verbal and written—are developing important skills for future learning and employment as well as strengthening their own understanding of mathematics” (Peck, p. 22).

Purposeful Practice

As concepts are learned, in order to be more readily accessed and applied, they must be connected to an existing network of ideas. Eventually, *some* of these new concepts may be consolidated into mathematical procedures or algorithms. *Purposeful practice* enables students to solidify their knowledge and ultimately to apply new concepts flexibly and procedures fluently to skilled performances. In turn, these new concepts and skills become part of the network of ideas that supports the development of still more complex mathematical reasoning. However, research shows that the premature introduction and practicing of procedures interferes with the development of conceptual understanding (Wearne & Hiebert, 1988). “Practice without understanding may be destructive. When students mimic a procedure without understanding it is difficult for them to go back later and build understanding” (Sutton, p. 81).

Classroom Environment

Research describes an environment in which mathematical sense-making is the “currency of the classroom,” intellectual risk-taking is encouraged and rewarded, and mathematical argumentation produces enhanced engagement and deep understanding (Hiebert, *p. 9*).

Technology and other learning tools are used to make sense of the mathematics rather than to bypass or undercut learning. Most importantly, challenging tasks are central to mathematics learning. “Worthwhile tasks should be intriguing, with a level of challenge that invites speculations and hard work” (National Council of Teachers of Mathematics, *p. 19*). Teachers’ probing questions guide students as they work together on carefully selected problems. Assessment is ongoing and both informs instruction and summarizes student understanding.

Implications for Classroom Instruction

These elements of effective math instruction are often organized within a lesson as the Launch, Exploration, Summary and Application or Assessment (LESA).

- *Launch*: During the launch, the teacher poses *but does not solve* a challenging problem that builds upon and extends students’ prior understanding.
- *Explore*: Within the exploration or investigation students often work together to make and test conjectures, offer explanations, and justify their own solutions.
- *Summary*: The teacher then has the responsibility to reconvene the class as a whole in order to make sure that students have, indeed, solidified their understanding of the lesson’s learning goal(s). This opportunity to share and summarize allows students to consider one another’s strategies, “debrief thinking processes” (EdThoughts, *p. 89*), and develop a shared understanding of the day’s mathematics.
- *Apply*: Following the summary, students should engage in purposeful practice in order to apply the new mathematical concepts. Ultimately, students should have opportunities to “transfer” these skills and understandings to novel problem situations and contexts.

The ability to use the language of mathematics to solve a range of challenging problems is, after all, the fundamental goal of mathematics instruction.

The Process Standards: Clarifying the Vision for K–12 Mathematics

The Process Standards are discussed first because they are an integral part of each of the Content Standards and can not be taught in isolation. In other words, to do the kind of work described in any of the Process Standards, students must also be doing the work described in at least one of the Content Standards. The following questions should be used to focus instruction on big mathematical ideas in K–12 classrooms:

- Is your plan working? Do you need to reconsider what you are doing?
- How are *solving* and *proving* different? How are *showing* and *explaining* different?
- How do you know when you have proven something?
- What does it take to verify a conjecture? How do you develop a convincing argument?
- How do you make sense of different strategies? How do you determine their strengths and weaknesses? How do you determine similarities and differences?
- Why do we classify? Why do we classify numbers? Why do we classify geometric objects? Why do we classify functions?

The Process Standards are also interrelated. Reasoning is the basis of problem solving while communication is necessary in order to make reasoning public. In 2000, NCTM added a standard on representation. Delaware chose to not add representation as a separate standard because of the prominent role representation plays within the Content Standards.

Whether in mathematics or in another discipline, problem solving is more than solving word problems—it involves work on tasks for which the solution path is not obvious. Well chosen tasks need not wait until students have mastered skills that may be involved in solving the problem, nor should they be “de-problematized” when students struggle with getting started. Well chosen tasks can be used to build new mathematics understandings and to extend a student’s mathematical knowledge. They may come from the student’s world, from pure mathematics, or from other content areas such as literature, science, or health.

Reasoning is a necessary component of knowing mathematics with understanding. Reasoning becomes a new “basic skill” when teaching for understanding becomes a goal of the mathematics program. Students of all ages should be encouraged to make and investigate

conjectures—informed predictions. In early grades, students reason about individual objects such as a number or a triangle. In upper elementary, they reason about a set or class of objects such as multiples of a given number or all triangles. In middle school, they reason about patterns and common attributes that can be generalized. Proof is reasoning made formal. As reasoning, making, testing, and justifying conjectures become a part of every mathematics classroom, proof becomes a habit of mind: a natural way of thinking mathematically.

Communicating mathematically involves speaking, writing, and listening—skills to express ideas and receive them. It makes mathematical thinking observable. Written communication may include pictures and drawings if they make mathematical thinking more clear to others. Students must have the critical listening skills in order to make sense of another’s strategy and to uncover the strengths and limitations of that strategy. In the early grades, student thinking will be expressed in “kid language.” Students should come to see a need for more precise language and eventually for commonly held definitions. The end result goes beyond memorizing definitions to the use of language to advance mathematical thinking.

We learn by building networks that connect new knowledge to existing knowledge (Donovan, 2005). When mathematics is learned as a series of isolated facts and procedures, a student’s understanding is not very deep and efforts to recall previously learned facts or procedures may be unproductive. The information may not be recalled or it may be muddled. So was it really learned? For school mathematics to be meaningful, connections must be made between the student’s intuitive, outside-of-school understanding of mathematics, and the concepts taught in school mathematics.

Ultimately, instruction in mathematics should help K–12 students develop the following understandings:

- Mathematics can be used to solve problems outside of the mathematics classroom.
- Mathematics is built on reason and always makes sense.
- Reasoning allows us to *make* conjectures and to *prove* conjectures.
- Classifying helps us build networks of mathematical ideas.
- Precise language helps us express mathematical ideas and receive them.

Numeric Reasoning: Clarifying the Vision for K–12 Mathematics

The following questions should be used to focus instruction on big numeric ideas in K–12 classrooms:

- What makes an estimate *reasonable*?
- What makes an answer *exact*?
- What makes a strategy both *effective* and *efficient*?
- What makes a solution *optimal*?

Young children develop a sense of number by counting objects. As numbers become larger, strategies such as count on, make ten, double, double plus one, and double minus one, deepen a student's sense of number and initiate the conceptual development of operations. These strategies lead to decomposing and recomposing numbers that are known to make numbers that are not known. The number line is expanded to represent numbers between counting numbers and numbers less than zero. Number sense includes recognizing that numbers can be represented in multiple formats that are equivalent. As the number system is uncovered from the use of counting numbers to fractions and decimals to irrational numbers, many of the same number properties apply but the use of those properties becomes more explicit.

The understanding of addition and subtraction emerges from decomposing and recomposing numbers. The concepts of addition and subtraction are fully developed with problems that can be classified as join, separate, part-part-whole, and compare. Join and separate problems have the same units. Part-part-whole problems have different but similar units that can be added or subtracted. While the differences may appear to be subtle, they represent very different thinking processes and students need to experience problems in all four classifications.

Multiplication begins as repeated addition and then evolves to a more efficient strategy using an array model or partial products. In high school, students will use this same strategy to multiply polynomials. Division is developed as repeated subtraction, as a fair way of sharing, as a fraction, and as a rate. Number sense enables a student's transition from additive reasoning to multiplicative reasoning to proportional reasoning.

Proportional reasoning is not the same as solving proportions by cross-multiplying. Students should not be introduced to cross-multiplication as a strategy to solve proportions until they have had sufficient symbolic algebra to understand *why* the strategy works. Conceptually proportional reasoning is about scaling up and scaling down and should be developed using models such as fraction strips, fraction bars, or rate/ratio tables.

Students should be encouraged to develop and utilize computational strategies that make sense to them *and* are effective. Sometimes students need assistance translating these initial effective strategies in to symbolic representations on paper. Over time, students should be encouraged to develop strategies that are both effective and efficient.

Even in classrooms that emphasize standard computational algorithms, students are not passive absorbers of knowledge. They will create their own strategies to solve problems. However, those strategies may come from faulty reasoning and lead to misconceptions when students have no opportunity to discuss and justify their strategies (Fuson, p 72).

Knowing what operation to use when is an important skill for students to acquire. It is equally important to know what tools to use when. Effective, efficient strategies that make sense to students will lead to better estimation and mental math skills—skills they can use any time. Students should learn to make smart choices about when to estimate an answer, when to use mental math, and when to use a calculator.

Ultimately, instruction in numeric reasoning should help K–12 students develop the following understandings:

- Numbers can be represented in multiple ways.
- The same operations can be applied in problem situations that seem quite different from one another.
- Being able to compute fluently means making smart choices about which tools to use and when to use them.
- Knowing the reasonableness of an answer comes from using good number sense and estimation strategies.

Algebraic Reasoning: Clarifying the Vision for K–12 Mathematics

The following questions should be used to focus instruction on big algebraic ideas in K–12 classrooms:

- How can change be described mathematically?
- How are patterns of change related to the behavior of functions?
- How do mathematical models/representations shape our understanding of mathematics?

Algebra is the study of the patterns and symbols that make up the underlying structure of our number system. Algebraic reasoning culminates in the development of a formalized symbolic language that emerges from an early understanding of the “doing and undoing” of number operations. As young children come to understand numbers, patterns, and symbols experientially, they develop the ability to think about numbers and symbols abstractly.

Very young children learn number primarily through experiences with counting. As this understanding of counting develops, children begin to use symbols to represent numbers and number sentences to represent operations. In these early years, the development of a symbolic language emerges from experiences with patterns. Students learn about number relationships such as doubles and halves, odds and evens, and multiples of twos, fives, and tens. These experiences with number patterns help students to come to the realization that numbers “behave” in predictable ways. Students represent these insights with number sentences; these number sentences form the beginning stages of symbolic reasoning.

In the intermediate grades, children look at patterns of growth and change. They make graphs and tables to look for trends and to make predictions. Algebraic reasoning is becoming more formal with students developing rules and formulas, such as the cost of a taxi ride or the area of a triangle. Students investigate how variables are related, translating number relationships into rules using variables. In order to apply these rules they solve simple equations using informal methods. Additional experiences with geometry and data reinforce this search for rules and generalizations.

By middle school, students are called upon to move fluently between a variety of models (tables, graphs, and symbolic rules) used to represent variable relationships. They are focused on the study of rates of change and the impact of slope in an equation, a graph, and a table. There is more emphasis on the proper use of symbolic language to describe variable operations. By eighth grade, students are introduced to algebraic transformations and manipulations; they are gaining experiences in demonstrating equivalence in algebraic expressions and generalizing from visual patterns and arithmetic sequences.

These students enter high school with a strong grasp of linear relationships, a formal understanding of the language of linear equations, slope and y-intercept, and a sense of how quadratic and exponential equations differ. High school students explore the increasingly formal, symbolic understandings of families of functions and symbolic transformations. They move fluently among symbolic and graphic representations with a more abstract reasoning system for expressing slope and finite differences and for demonstrating algebraic equivalence. They reason about families of functions and interpret symbolic generalizations about the behavior of standard forms. Symbolic algebra becomes the language of transformation and reasoning used in problem solving and for justification and proof in each of the four strands.

Ultimately, instruction in algebraic reasoning should help K–12 students develop the following understandings:

- Change is fundamental to understanding functions.
- Numbers or objects that repeat in predictable ways can be described or generalized.
- An operation can be “undone” by its inverse.
- Rules of arithmetic and algebra can be used together with notions of equivalence to transform equations and inequalities so solutions can be found.

Geometric Reasoning: Clarifying the Vision for K–12 Mathematics

The following questions should be used to focus instruction on big geometric ideas in K–12 classrooms:

- How are measurement and counting related?
- How does *what* we measure affect *how* we measure? How can space be defined through numbers/measurement?
- Why do we compare contrast, and classify objects?
- How do decomposing and recomposing shapes help us build our understanding of mathematics?
- How can transformations be described mathematically?

Geometry (derived from “geo” and “metric” so, literally, the “measure of the earth”) is comprised of several important but distinct strands, all of which should be developed across grades K–12. In very young children, we build upon intuitive understandings that, in combination, make up the sense of space. These begin with simple ideas of relative position (in front of, “over there”) and culminate with two- and then three-dimensional coordinate geometry. Gradually, distance measurement becomes an important component of this spatial sense.

Early on, movement in both two and three dimensions becomes another component of a child’s developing spatial sense. This movement is first formalized as simple “transformations” of position accessible in the primary grades—flips, slides, and turns—and then given more formal definition and deeper exploration in the middle grades as reflections, translations, and rotations. Ultimately, transformations of position, size, and shape are embedded within the coordinate plane and described algebraically using matrices.

Visual representations play a pivotal role in supporting reasoning about geometric objects and properties. Because most representational systems, from a “flat” piece of paper to a computer screen, are two-dimensional, conventions for representing three-dimensional objects in two dimensions, “projections,” must be developed. A range of representational systems from paper folding (useful for emphasizing instances of congruence, for example) to Geo Logo (students must input side length and angle measure) to computer-based geometric “supposers,” which support the making and testing of conjectures about geometric figures, find application across the grades.

Natural three-dimensional objects can be abstracted into perfect forms, tree trunks as the prototypes of the cylinder, the arc of an ancient slingshot whirling overhead as the ideal of the circle with a given center and radius. These idealized geometrical forms are then categorized, their elements defined, and their relationship to one another determined. The classification of two- and three-dimensional geometric objects proceeds from an essentially perceptual and inductive basis—rectangles “look the same”—to an understanding of the set of characteristics that these objects share, for example, all rectangles have four right angles and opposite sides congruent and parallel. A primarily inductive reasoning about geometric characteristics transitions into a more deductive understanding of how one class of objects relates to another, for example, a square is a rectangle with an added constraint.

The development of a fully axiomatic system for the classification of geometric objects provides an important opportunity for high school students to develop deductive reasoning and proof.

Ultimately, instruction in geometric reasoning should help K–12 students develop the following understandings:

- Two- and three-dimensional objects can be described, classified, and analyzed by their attributes.
- An object in a plane or in space can be oriented in an infinite number of ways while maintaining its size or shape.
- An objects' location on a plane or in space can be described quantitatively.
- Linear measure, area, and volume are fundamentally different but may be related to one another in ways that permit calculation of one given the other.

Quantitative Reasoning: Clarifying the Vision for K–12 Mathematics

The following questions should be used to focus instruction on big ideas in data and probability in K–12 classrooms:

- What is average?
- What makes a data representation useful?
- How does my sample affect confidence in my predication?
- What is fair?

In the primary grades, students collect data about important questions that involve choices, “What do you like better, whales or eagles, pizza or ice cream?” They organize the results in pictographs with visuals that physically represent the choices (a picture of a whale or an ice cream cone is placed on the graph and represents the choice). Students construct their own data displays and use them to answer questions such as “which is less and which is more.” As the study of data progresses, students conduct experiments and surveys that include more categories and choices. Students are called upon to order their data using tables and tallies, and ultimately to represent both categorical and numerical data. Their use of graphs expands to include dot/line plots, and they begin to think more globally about the shape of the distribution of the data. Student experiences with probability are also based on common experiences and shared understandings of the world around them such as “how likely is it that the sun will shine tomorrow or is it likely that we will have rain tomorrow?” Their thinking about probability is categorical (likely vs. unlikely or possible vs. impossible) rather than numerical.

As they enter the intermediate grades, students begin to think more critically about one-variable data sets. They grapple with the need to determine a typical, or average, value from a set of data, which leads to understanding different measures of central tendency. They pose questions and design experiments that require the collection of numerical data. In addition to pictographs, bar graphs, and line plots, they begin to use circle graphs to represent and analyze data. Student experiences with probability expand and become more formalized as they conduct experiments to determine whether a game with spinners or number cubes is “fair.” They develop ways to make organized lists and sample spaces to represent all possible

outcomes, primarily focused on fair games, and use statistical tools such as tallies and relative frequencies to solve probability problems.

In middle school, students are able to take two-variable data sets and compare those using measures of central tendency, stem and leaf plots, histograms and box and whisker plots. They can demonstrate that there are uses and misuses of data and how measures of central tendency may be misleading. Students are more aware of the impact of scaling on graphs and can demonstrate their understanding by building effective displays of data. They use scatter plots to investigate the relationship between two variables and use lines of best fit, where appropriate, to make predictions.

Middle school experiences with probability become more sophisticated and involve the use of simulations. Students think about how to select a “fair” (random) sample. They develop an understanding of theoretical probability and how it relates to experimental probability. Well-organized tree diagrams and lists are used to justify calculations of theoretical probabilities. Conceptual and procedural knowledge of probability now requires fluency with rational number and a strong grounding in proportional reasoning.

In high school, while conceptual foundations for both statistics and probability continue to be developed using data drawn from experiments, students are expected to generalize and formalize their knowledge. Scatter plots and mathematical models enable predictions to be made, while mathematical reasoning addresses possible limitations for the predictions. Experimental probabilities support conjectures regarding theoretical probabilities, which can now be computed using formulas. Statistics and probability are woven together through investigations of distributions of data, including normal distributions.

Ultimately, instruction in quantitative reasoning should help K-12 students develop the following understandings:

- The question to be answered determines the data to be collected and how best to collect it.
- Basic statistical techniques can be used to analyze data in the workplace
- The probability of an event can be used to predict the probability of future events

Resources

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